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QUALITATIVE PROPERTIES OF OPTIMAL POLICIES FOR SEMI-SEPARABLE MARKOV DECISION PROCESSES

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I. Introduction

A previous paper (Mendelssohn [4]) shows how to reduce the computational effort involved in solving "semi-separable" Markov decision processes (MDP's). Semi-separable MDP's often arise as discrete approximations to the problem of finding f(x) such that:

$$f(x) = \max\left\{G(x, y) + \alpha E\left(f(s[y, D])\right) : y \in Y(x);\ x \in X\right\}$$

$$0 \leq \alpha < 1 \tag{1}$$

Problems of this form arise in the context of harvesting problems, capital accumulation and consumption, reservoir management, and other areas of stochastic optimization. The main characteristic is that the transition probabilities depend only on the decision y, and not on the state x. As such, semi-separable processes are an extension of the "separable" MDP's discussed by Denardo [2], where it is further assumed that G(x, y) = a(x) + b(y).

If it is assumed that $Y(x) = \{y : 0 \leq y \leq x\}$, Mendelssohn [4] shows that only two decisions can be optimal at any state x. Let A(x) be an optimal policy function. Let G_x^y (x = 0, 1, ..., n; y = 0, 1, ..., x), be a discretization of G(x, y). Define b_j^* as:

$$b_j^* = \max_{0 \leq i \leq j}\left(G_j^i - G_{j-1}^i\right) \qquad j = 1, 2, \ldots, n$$

and let i_j^* be the argument where b_j^* obtains its maximum. Then [4]:

$$A(x) = x \text{ or } i_x^*, \qquad x = 0, 1, \ldots, n. \tag{2}$$

An interesting feature of (2) is that strong qualitative properties of an optimal policy depend only on properties of the one-period return function, and not on the transition probabilities. This makes it possible to generalize results in specific contexts that have not proven amenable to other forms of analysis.

II. Qualitative Results

A model of interest in capital accumulation, economic growth and managing renewable resources (see Brock and Mirman [1], Mirman and Zilcha [6], Mendelssohn and Sobel [5], and references cited in these papers) assumes that:

(i) $G(x, y) = G(x-y)$; $G(\cdot)$ concave, nondecreasing and continuous

(ii) $s[\cdot, D]$ concave and continuous.

Usually it is also assumed that $s[\cdot, D]$ is nondecreasing. It is then shown that for a continuous state model:

$$0 \leq \frac{d}{dx} A(x) \leq 1 \qquad (3)$$

or for a discrete state model:

$$A(x+1) = \text{either } A(x) \text{ or } A(x) + 1. \qquad (4)$$

Theorem 2.1 generalizes these results by only requiring that $G(\cdot)$ be concave, and no assumptions on $s[\cdot, \cdot]$. The result is for the discrete state space model.

Theorem 2.1 In the discrete version of (1), assume $G(x, y) = G(x-y)$, $G(\cdot)$ *concave and not separable. Then there exist* j *numbers:*

$$0 \le x_1 \le x_2 \le x_3 \cdots \le x_j \le n$$

such that:

$$A(x) = \begin{cases} x & 0 \le x \le x_1 \\ x-1 & x_i < x \le x_{i+1}; \quad i \text{ odd} \\ x & x_i < x \le x_{i+1}; \quad i \text{ even} \end{cases}$$

$$i = 0, 1, \ldots, j$$

Proof. $G(\cdot)$ is assumed to be concave. This implies $i_j^* = j-1$, so that $b_j^* = b^* = G(1) - G(0)$. The result is then straightforward. □

If $G(x-y) = p \cdot (x-y)$, then i_x^* could be any $y < x$. However, the separable form of $p \cdot (x-y)$ provides a straightforward modification to theorem 2.1.

Corollary 2.1 If under the conditions of theorem 2.1, $G(x-y) = p \cdot (x-y)$, *then there exist* j *integers:*

$$0 \le x_1 \le x_2 \le x_3 \le \cdots \le x_j \le n$$

such that

$$\begin{cases} x & 0 \le x < x_1 \\ x_i & x_i \le x < x_{i+1} \quad i \text{ odd} \\ x & x_i \le x < x_{i+1} \quad i \text{ even} \end{cases}$$

$$i = 0, 1, \ldots, n$$

In corollary 2.1, the x_i, i odd, are the discrete equivalent of local maxima of:

$$\alpha pE\left\{f(s[y, D])\right\} - p \cdot y$$

and the x_i's, i even are the smallest points on the interval (x_i, x_{i+2}), i odd, such that:

$$\left[\alpha pE\left\{f(s[x_{i+2}, D])\right\} - p \cdot x_{i+2}\right] - \left[\alpha pE\left\{f(s[x_i, D])\right\} - px_i\right] > 0$$

and

$$\left[\alpha pE\left\{f(s[x_{i+1}, D])\right\} - p \cdot x_{i+1}\right] - \left[\alpha pE\left\{f(s[x_i, D])\right\} - px_i\right] > 0$$

III. Computational Results

For some problems, it is known a priori that if $A(x) \neq x$, then $A(x+1) \neq x+1$. In the context of theorem 2.1, this is true if it is assumed that $s[\cdot, D]$ is concave (Mendelssohn and Sobel [5]). It may also be known, say from the two-period approximation to the infinite horizon problem, that:

$$A(x) = \begin{cases} x & \text{for } x \leq j \\ x-1 & \text{for } x \geq j^u \end{cases} \qquad (5)$$

The linear program that solves this problem is:

$$\text{minimize} \sum_{x=0}^{n} f_x \tag{6}$$

$$\text{s.t.} \quad \sum_{x=0}^{n} \left(\delta_{xj} - \alpha P_x^j\right) f_x \geq G(0) \qquad j = 0, 1, \ldots, n \tag{6a}$$

$$\sum_{x=0}^{n} \left(\delta_{x,\, j-1} - \alpha P_x^{j-1}\right) f_x \geq G(1) \qquad j = 0, 1, \ldots, n \tag{6b}$$

where $\delta_{x,\, j} = \begin{cases} 0 & x \neq j \\ 1 & x = j \end{cases}$

From (5), it follows that equality must hold in (6a) for $0 \leq j \leq j^L$, and equality must hold in (6b) for $j^u \leq j \leq n$. These variables can be solved for in terms of $f_{j^L}, f_{j^L+1}, \ldots, f_{j^u-1}$, so that the remaining LP can be greatly reduced in size. The new LP can also be solved by iterative methods such as successive approximations with greatly reduced computational effort.

It should be noted, however, that even if j^L, j^u are not known a priori, a sparse representation of (6) is possible. Transform variables by:

$$f_i = \sum_{j=0}^{i} v_i \qquad i = 0, 1, \ldots, n$$

and transform (6) into equality constraints by subtracting surplus variables. This yields:

$$\text{minimize} \sum_{x=0}^{n} (n+1-x)v_x \tag{7}$$

$$\text{s.t.} \quad \sum_{x=0}^{n} \left(\delta_{x \leq j} - \alpha \sum_{i=x}^{n} P_i^j\right) v_j - \lambda_j^j = G(0) \qquad j = 0, 1, \ldots, n \tag{7a}$$

$$v_{j+1} + \lambda_j^j - \lambda_{j+1}^j = G(1) \qquad j = 0, 1, \ldots, n-1 \tag{7b}$$

$$\lambda_j^i \geq 0 \qquad j = 0, 1, \ldots, n$$
$$i = j, j-1$$

where $\delta_{x \leq j} = \begin{cases} 1 & x \leq j \\ 0 & x > j \end{cases}$

Besides being a much sparser LP, (7) has several other advantages over (6). The expectation of a nonnegative random variable x with distribution function F(x) can be expressed as:

$$E(x) = \int_0^n [1 - F(x)]dx$$

(Karlin and Taylor [3]), where $P\{x \leq n\} = 1$. Note that the rows (7a) are the discrete equivalent of:

$$\int_0^n \left(\delta_{x \leq j} - \alpha \int_x^n s[j, \xi]dF(\xi)\right) v(x)dx \tag{8}$$

After choosing a grid to discretize (1), there are several ways of discretizing the transition probabilities. The LP (7) suggests that for this problem an efficient method may be to set:

$$\delta_{x \leq j} - \alpha \sum_{i=x}^{n} P_i^j = \delta_{x \leq j} - \alpha \int_x^n s[j, \xi]dF(\xi) \tag{9}$$

As integration tends to "smooth out" functions, (9) should give a better approximation than by using (6) and letting

$$p_x^j = \int_x^{x+1} s[j, \xi]dF(\xi)$$

or some other approximation to the transition probabilities themselves.

III. Summary

In this paper, it has been assumed that x, y are scalars. However, as pointed out in [4], similar results can be derived for x, y vectors. Thus, related qualitative and computational results can be derived for the vector case. As such, semi-separable MDP's cover a broad class of problems that can be realistically computed. LP's similar to (7) also lend themselves to row or column aggregation such as in Zipkin [7], which can be used to obtain "good" policies for quite large problems.

[1] Brock, W. A. and Mirman, L. J., "Optimal Economic Growth and Uncertainty: The Discounted Case," J. Econ. Theory IV (1972):479-513.

[2] Denardo, E. V., "Separable Markovian Decision Problems," Management Science 14(1968):451-462.

[3] Karlin, S. and Taylor, H. M., A First Course in Stochastic Processes. 2nd Ed. Academic Press, N.Y., 557 p.

[4] Mendelssohn, R., "Increasing Computational Efficiency for Semi-Separable Markov Decision Processes," U.S. Dep. of Commer, Natl. Mar. Fish. Serv., SWFC Admin. Rep. 2H (1978), 10 p.

[5] Mendelssohn, R. and Sobel, M. J., "Capital Accumulation and the Optimization of Renewable Resource Models." Submitted to J. Econ. Theory (1977).

[6] Mirman, L. J. and Zilcha, I., "On Optimal Growth Under Uncertainty," J. Econ. Theory VII (1975):329-339.

[7] Zipkin, P. H., "A Priori Bounds for Aggregated Linear Programs With Fixed-Weight Disaggregation," Technical Report #86, School of Organization and Management, Yale University (1977), 37 p.