

WHEN THE TWO-PERIOD OPTIMAL POLICY IS OPTIMAL  
OVER AN INFINITE HORIZON: A NOTE

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DRAFT FOR COMMENT

Problems arising in managing renewable resources, particularly for yield, often take the form:

$$\max E \left\{ \sum_{t=1}^T \alpha^{t-1} P \cdot (x_t - y_t) \right\}$$

$$\text{s.t. } x_{t+1} = s[y_t, D_t]$$

$$0 \leq y_t \leq x_t$$

where  $x$  is the state,  $y$  is the decision, and the  $D_i$ 's are independent, identically distributed random variables. The planning horizon  $T$  may be either finite or infinite.

If  $T$  is finite, Mendelsohn and Sobel (1977) derive the finite dynamic program:

$$f_0(\cdot) \equiv 0$$

$$f_n(x) = \max_{0 \leq y \leq x} \left\{ P \cdot (x-y) + \alpha E f_{n-1}(s[y, D]) \right\} \quad (1)$$

and over an infinite horizon:

$$f(x) = \max_{0 \leq y \leq x} \left\{ P \cdot (x-y) + \alpha E f(s[y, D]) \right\} \quad (2)$$

In a series of recent papers (Mendelsohn 1978a, b, c) I show how to greatly reduce the effort involved in solving (2). In this paper, I show that for a special case, an optimal policy in (1) for  $n = 2$  is optimal in (2).

Consider the following assumption:

- (i)  $f(\cdot)$  is concave, continuous
- (ii)  $E_s[y, D]$  is unimodal and differentiable with respect to  $y$ .

Conditions which are sufficient for (ii) to be valid are given in Mendelssohn and Sobel (1977). Let  $y_2^*$  be the solution to the following equation:

$$E\left\{s^{[1]}[y, D]\right\} = \frac{1}{\alpha} \quad (3)$$

Mendelssohn and Sobel (1977) show that both (1) and (2) have a base stock policy as an optimal policy. That is, there is a  $y^*$  such that an optimal policy is to choose:

$$\text{minimum } (x, y^*)$$

Clearly  $y_2^*$  is the base stock size at  $n = 2$ . Theorem 1 proves  $y_2^*$  is the base stock size in (2) also.

Theorem 1. Assumptions (i)-(ii) imply  $y_2^*$  is an optimal base stock size in (2).

Proof. It is straightforward to show that at  $y^*$ ,

$$E\left\{s[y^*, D]\right\} \geq y^*$$

(see, for example, Mendelssohn and Sobel 1977).

From equation (1), this implies for  $w = E\{s[y^*, D]\}$ :

$$f(w) = p \cdot (w - y^*) + \alpha E f(s[y^*, D])$$

Applying Jensen's inequality yields:

$$p \cdot (w - y^*) + \alpha E f(s[y^*, D]) \leq p \cdot (w - y^*) + \alpha f(Es[y^*, D])$$

which implies:

$$p \cdot (w - y^*) + \alpha E f(s[y^*, D]) \leq p \cdot (w - y^*) + \alpha f(w) = p \cdot (w - y^*) + \alpha E f(s[y^*, D]) \quad (4)$$

Equation (4) implies at  $y^*$ ,  $\alpha E f(s[y^*, D]) = \alpha f(Es[y^*, D])$ .

At  $y = y_2^*$ ,  $p \cdot (w - y) + \alpha f(w)$  achieves a maximum, which implies  $p \cdot (w - y) + \alpha E f(s[y, D])$  also achieves a maximum at  $y_2^*$ . Since a base stock policy is optimal,  $y^* = y_2^*$  is the base stock size.

□

What is convenient about theorem 1 is that equation (3) can be solved on nothing more than a pocket calculator. It also underlines a very real problem in using expected value as a criterion for optimization. That is,  $y_2^*$  is optimal no matter what the variance of  $D$ , so long as the expectation on  $D$  is the same. This suggests that when going from deterministic to stochastic models, the expectation of the deterministic objective most likely is not the proper objective function for the stochastic model. There are several ways around this problem. The first is to use utility theory or other related methods to determine the decisionmaker's attitude towards risk. The second is to include smoothing costs of the form:

$$\begin{aligned} &\varepsilon \left( (x_{t-1} - y_{t-1}) - (x_t - y_t) \right) && \text{if } (x_{t-1} - y_{t-1}) > (x_t - y_t) \\ &\gamma \left( (x_t - y_t) - (x_{t-1} - y_{t-1}) \right) && \text{if } (x_t - y_t) > (x_{t-1} - y_{t-1}) \end{aligned}$$

In a future paper, I will show that for  $\varepsilon = \gamma$ , this is equivalent to weighting the mean return against the variance of the return. By parameterizing on  $\varepsilon$  (therefore  $\gamma$ ), it is then possible to explore the mean-variance tradeoff.

I suspect, but have not been able to prove, that if assumptions (i)-(ii) are valid, then a two-period optimal policy to the smoothing cost problem is a good approximation to a true infinite horizon optimal policy. This will be explored numerically.

References

- Mendelssohn, R. (1978a) Increasing computational efficiency for semi-separable Markov decision processes. U.S. Dep. Commer., Natl. Mar. Fish. Serv., SWFC Admin. Rep. 2H, 1978, 16 p.
- \_\_\_\_\_. (1978b) Recovering optimal policies and qualitative properties of optimal policies for semi-separable Markov decision processes. U.S. Dep. Commer., Natl. Mar. Fish. Serv., SWFC Admin. Rep. 3H, 1978, 7 p.
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- Mendelssohn, R. and M. J. Sobel (1977) Capital accumulation and the optimization of renewable resource models. Submitted to J. Econ. Theory.



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Dr. Matthew J. Sobel  
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Dear Matt:

Enclosed are the revised versions of the papers I sent you. Also included are several new papers which extend these results in specific instances. Your comments, corrections, questions, etc., will be appreciated. If you have some good applications for this stuff let me know. I am trying to find just how well some of this performs.

Keep in touch--spring will come sooner than you know.

With regards,

Roy Mendelssohn  
Operations Research Analyst

Enclosure (SWFC Admin. Rep. 2H, 3H, 4H, 9H, 10H, 1978)

RM:ey  
cc: Mendelssohn  
HL

Identical letter to:

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